

INFINITE-DIMENSIONAL ALGEBRAS IN DIMENSIONALLY REDUCED STRING THEORY

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Abstract

We examine 4-dimensional string backgrounds compactified over a two torus. There exist two alternative effective Lagrangians containing each two $SL(2)/U(1)$ sigma-models. Two of these sigma-models are the complex and Kähler structures on the torus. The effective Lagrangians are invariant under two different $O(2,2)$ groups and by the successive applications of these groups the affine $\hat{O}(2,2)$ Kac-Moody algebra is emerged. The latter has also a non-zero central term which generates constant Weyl rescalings of the reduced 2-dimensional background. In addition, there exists a number of discrete symmetries relating the field content of the reduced effective Lagrangians.

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It is known that higher-dimensional gravitational theories exhibit unexpected new symmetries upon reduction [1]. Dimensional reduction of the string background equations [2] with dilaton and antisymmetric field also exhibit new symmetries as for example dualities [3]–[5]. However, the exact string symmetries will necessarily be subgroups or discrete versions of the full symmetry group of the string background equations and thus, a study of the latter would be useful. The empirical rule is that the rank of the symmetry group increases by one as the dimension of the space-time is decreased by one after dimensional reduction [6]. However, the appearance of non-local currents in two-dimensions in addition to the local ones, turns the symmetry group infinite dimensional. Let us recall the $O(8, 24)$ group of the heterotic string after reduction to three dimensions [7] which turns out to be the affine $\hat{O}(8, 24)$ algebra by further reduction to two dimensions [8] or the $\hat{O}(2, 2)$ algebra after the reduction of 4-dimensional backgrounds [9]. The latter generalizes the Geroch group of Einstein gravity [10]–[12]. We will examine here the “affinization” of the symmetry group of the string background equations for 4-dimensional space-times with two commuting Killing vectors and we will show the emergence of a central term. Generalization to higher dimensions is straightforward.

The Geroch group is the symmetry group which acts on the space of solutions of the Einstein equations [10]. Its counterpart in string theory, the “string Geroch group”, acts, in full analogy, on the space of solutions of the one-loop beta functions equations [9]. The Geroch group, as well as its string counterpart, results by dimensional reducing four-dimensional backgrounds with zero cosmological constant over two commuting, orthogonal transitive, Killing vectors or, in other words, by compactifying M_4 to $M_2 \times T^2$. In dimensional reduced Einstein gravity, there exist two $SL(2, \mathbb{R})$ groups (the Ehlers’ and the Matzner-Misner groups [13]) acting on the space of solutions, the interplay of which produce the infinite dimensional Geroch group. In the string case, we will see that apart from the Ehlers and Matzner-Misner groups acting on the pure gravitational sector, there also exist two other $SL(2, \mathbb{R})$ groups, one of which generates the familiar S-duality, acting on the antisymmetric-dilaton fields sector.

The Geroch group was also studied in the Kaluza-Klein reduction of supergravity theories [1]. It was B. Julia who showed that the Lie algebra of the Geroch group in Einstein gravity is the affine Kac-Moody algebra $\hat{sl}(2)$ and he pointed out the existence of a central term [13]. We will show here that in the string case, after the reduction to $M_2 \times T^2$, there exist four $SL(2, \mathbb{R})$ groups, the interplay of which produce the infinite dimensional Geroch group. However, there is also a central term which rescales the metric of M_2 so that the Lie algebra of the string Geroch group turns out to be the $\hat{sl}(2) \times \hat{sl}(2) \simeq \hat{o}(2, 2)$ affine

Kac-Moody algebra. The appearance of a non-zero central term already at the tree-level is rather surprising since usually such terms arise as a consequence of quantization [15]. Here however, the central term acts non-trivially even at the “classical level” by constant Weyl rescalings of the reduced two-dimensional space M_2 .

String propagation in a critical background \mathcal{M} , parametrized with coordinates (x^M) and metric $G_{MN}(x^M)$, is described by a two-dimensional sigma-model action

$$S = \frac{1}{4\pi\alpha'} \int d^2z (G_{MN} + B_{MN}) \partial x^M \bar{\partial} x^N - \frac{1}{8\pi} \int d^2z \phi R^{(2)}, \quad (1)$$

where B_{MN} , ϕ are the antisymmetric and dilaton fields, respectively. The conditions for conformal invariance at the 1-loop level in the coupling constant α' are

$$\begin{aligned} R_{MN} - \frac{1}{4} H_{MK\Lambda} H_N^{K\Lambda} - \nabla_M \nabla_N \phi &= 0 \\ \nabla^M (e^\phi H_{MNK}) &= 0 \\ -R + \frac{1}{12} H_{MNK} H^{MNK} + 2\nabla^2 \phi + (\partial_M \phi)^2 &= 0, \end{aligned} \quad (2)$$

and the above equations may be derived from the Lagrangian [16]

$$\mathcal{L} = \sqrt{-G} e^\phi \left(R - \frac{1}{12} H_{MNK} H^{MNK} + \partial_M \phi \partial^M \phi \right), \quad (3)$$

where $H_{MNA} = \partial_M B_{NA} + \text{cycl. perm.}$ is the field strength of the antisymmetric tensor field B_{MN} .

The right-hand side of the last equation in eq. (2) has been set to zero assuming that the central charge deficit δc is of order α'^2 (no cosmological constant). We will also assume that the string propagates in $M_4 \times K$ with $c(M_4) = 4 + \mathcal{O}(\alpha'^2)$ and that the dynamics is completely determined by M_4 while the dynamics of the internal space K is irrelevant for our purposes. Thus, we will discuss below general 4-dimensional curved backgrounds in which $H_{\mu\nu\rho}$ can always be expressed as the dual of H^M

$$H_{MNA} = \frac{1}{2} \sqrt{-G} \eta_{MNAK} H^K, \quad (4)$$

with $\eta_{1234} = +1$ and $M, N, \dots = 0, 1, 2, 3$. The Bianchi identity $\partial_{[K} H_{MNA]} = 0$ gives the constraint

$$\nabla_M H^M = 0, \quad (5)$$

which can be incorporated into (3) as $b \nabla_M H^M$ by employing the Lagrange multiplier b so that (3) turns out to be

$$\mathcal{L} = \sqrt{-G} e^\phi \left(R - \frac{1}{2} s H_M H^M + \epsilon^{-\phi} b \nabla_M H^M + \partial_M \phi \partial^M \phi \right). \quad (6)$$

$s = \pm 1$ for spaces of Euclidean or Lorentzian signature, respectively and we will assume that $s = -1$ since the results may easily be generalized to include the $s = +1$ case as well. We may now eliminate H_M by using its equation of motion

$$H_M = e^{-\phi} \partial_M b, \quad (7)$$

and the Lagrangian (6) turns out to be

$$\mathcal{L} = \sqrt{-G} e^\phi \left(R - \frac{1}{2} e^{-2\phi} \partial_M b \partial^M b + \partial_M \phi \partial^M \phi \right). \quad (8)$$

Let us now suppose that the space-time M_4 has an abelian space-like isometry generated by the Killing vector $\xi_1 = \frac{\partial}{\partial \theta_1}$ so that the metric can be written as

$$ds^2 = G_{11} d\theta_1^2 + 2G_{1\mu} d\theta_1 dx^\mu + G_{\mu\nu} dx^\mu dx^\nu, \quad (9)$$

where $\mu, \nu, \dots = 0, 2, 3$ and $G_{11}, G_{1\mu}, G_{\mu\nu}$ are functions of x^μ . We may express the metric (9) as

$$ds^2 = G_{11} (d\theta_1 + 2A_\mu dx^\mu)^2 + \gamma_{\mu\nu} dx^\mu dx^\nu, \quad (10)$$

where

$$\begin{aligned} \gamma_{\mu\nu} &= G_{\mu\nu} - \frac{G_{1\mu} G_{1\nu}}{G_{11}}, \\ A_\mu &= \frac{G_{1\mu}}{G_{11}}. \end{aligned} \quad (11)$$

The metric (10) indicates the $M_3 \times S^1$ topology of M_4 and $\gamma_{\mu\nu}$ may be considered as the metric of the 3-dimensional space M_3 . Space-times of this form have extensively been studied in the Kaluza-Klein reduction where A_μ is considered as a U(1)-gauge field. The scalar curvature R for the metric (10) turns out to be

$$R = R(\gamma) - \frac{1}{4} G_{11} F_{\mu\nu} F^{\mu\nu} - \frac{2}{G_{11}^{1/2}} \nabla^2 G_{11}^{1/2}, \quad (12)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\nabla^2 = \frac{1}{\sqrt{-\gamma}} \partial_\mu \sqrt{-\gamma} \gamma^{\mu\nu} \partial_\nu$. By replacing (12) into (3) and integrating by parts we get the reduced Lagrangian

$$\mathcal{L} = \sqrt{-\gamma} G_{11}^{1/2} e^\phi \left(R(\gamma) - \frac{1}{4} G_{11} F_{\mu\nu} F^{\mu\nu} + \frac{1}{G_{11}} \partial_\mu G_{11} \partial^\mu \phi - \frac{1}{4} \frac{1}{G_{11}} H_{\mu\nu} H^{\mu\nu} + \partial_\mu \phi \partial^\mu \phi \right) \quad (13)$$

where $H_{\mu\nu} = H_{\mu\nu 1} = \partial_\mu B_{\nu 1} - \partial_\nu B_{\mu 1}$. (A general discussion on the dimensional reduction of various tensor fields can be found in [17]). We have taken $H_{\mu\nu\rho} = 0$ since in three dimensions

$B_{\mu\nu}$ has no physical degrees of freedom. Let us note that the Lagrangian (13) is invariant under the transformation

$$\begin{aligned} G_{11} &\rightarrow \frac{1}{G_{11}}, \\ H_{\mu\nu} &\rightarrow F_{\mu\nu}, \\ \phi &\rightarrow \phi - \ln G_{11}, \\ \gamma_{\mu\nu} &\rightarrow \gamma_{\mu\nu}, \end{aligned} \tag{14}$$

which, in terms of G_{11} , $G_{1\mu}$, $G_{\mu\nu}$, $B_{1\mu}$ and ϕ may be written as

$$\begin{aligned} G_{11} &\rightarrow \frac{1}{G_{11}}, & B_{\mu 1} &\rightarrow \frac{G_{\mu 1}}{G_{11}}, \\ G_{1\mu} &\rightarrow \frac{B_{\mu 1}}{G_{11}}, & G_{\mu\nu} &\rightarrow G_{\mu\nu} - \frac{G_{1\mu}^2 - B_{\mu 1}^2}{G_{11}}, \\ \phi &\rightarrow \phi - \ln G_{11}, \end{aligned} \tag{15}$$

and it is easily be recognized as the abelian duality transformation.

Let us further assume that M_3 has also an abelian spece-like isometry generated by $\xi_2 = \frac{\partial}{\partial \theta_2}$ so that $M_3 = M_2 \times S^1$. We will further assume that the two Killings (ξ_1, ξ_2) of M_4 are orthogonal to the surface M_2 . Thus, the metric (9) can be written as

$$ds^2 = G_{11}d\theta_1^2 + 2G_{12}d\theta_1d\theta_2 + G_{22}d\theta_2^2 + G_{ij}dx^i dx^j, \tag{16}$$

where $i, j, \dots = 0, 3$ and G_{11} , G_{12} , G_{22} , G_{ij} are functions of x^i only. We may write the metric above as

$$ds^2 = G_{11}(d\theta_1 + Ad\theta_2)^2 + Vd\theta_2^2 + G_{ij}dx^i dx^j, \tag{17}$$

where

$$A = \frac{G_{12}}{G_{11}}, \quad V = \frac{G_{11}G_{22} - G_{12}^2}{G_{11}}. \tag{18}$$

By further reducing (13) with respect to ξ_2 and using the fact that the only non-vanishing components of $F_{\mu\nu}$ and $H_{\mu\nu}$ are

$$\begin{aligned} F_{i2} &= \partial_i A, \\ H_{i2} &= \partial_i B, \end{aligned} \tag{19}$$

with $B = B_{21}$, we get

$$\begin{aligned} \mathcal{L} &= \sqrt{-G^{(2)}G_{11}}^{1/2} V^{1/2} e^\phi \left(R(G^{(2)}) - \frac{1}{2}(\partial A)^2 \frac{G_{11}}{V} - \frac{1}{8}(\partial \ln \frac{G_{11}}{V})^2 \right. \\ &\quad \left. - \frac{1}{2}(\partial B)^2 \frac{1}{G_{11}V} - \frac{1}{8}(\partial \ln G_{11}V)^2 + (\partial \tilde{\phi})^2 \right), \end{aligned} \tag{20}$$

where $\tilde{\phi} = \phi + \frac{1}{2} \ln G_{11} V$ and $(\partial\phi)^2 = \partial_i \phi \partial^i \phi$. Let us now introduce the two complex coordinates τ, ρ [18] defined by

$$\tau = \tau_1 + i\tau_2 = \frac{G_{12}}{G_{11}} + i \frac{\sqrt{G}}{G_{11}}, \quad (21)$$

$$\rho = \rho_1 + i\rho_2 = B_{21} + i\sqrt{G}, \quad (22)$$

where $G = G_{11}G_{22} - G_{12}^2$ is the determinant of the metric on the 2-torus $T^2 = S^1 \times S^1$, so that τ, ρ turn out to be the complex and Kähler structure on T^2 . In terms of τ, ρ , the Lagrangian (20) is written as

$$\mathcal{L} = \sqrt{-G^{(2)}} e^{\tilde{\phi}} \left(R(G^{(2)}) + 2 \frac{\partial\tau\partial\bar{\tau}}{(\tau - \bar{\tau})^2} + 2 \frac{\partial\rho\partial\bar{\rho}}{(\rho - \bar{\rho})^2} + (\partial\tilde{\phi})^2 \right), \quad (23)$$

where $R(G^{(2)})$ is the curvature scalar of M_2 . The Lagrangian above is clearly invariant under the $SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \simeq O(2, 2, \mathbb{R})$ transformation

$$\begin{aligned} \tau &\rightarrow \tau' = \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1, \\ \rho &\rightarrow \rho' = \frac{\alpha\rho + \beta}{\gamma\rho + \delta}, \quad \alpha\delta - \gamma\beta = 1. \end{aligned} \quad (24)$$

There also exist discrete symmetries acting on the (τ, ρ) space which leave $\tilde{\phi}$ invariant. One of these interchanges the complex and Kähler structures

$$D: \quad \tau \leftrightarrow \rho, \quad \tilde{\phi} \rightarrow \tilde{\phi}. \quad (25)$$

In terms of the fields G_{11}, G_{12}, G_{22} , and B_{12} the above transformation is written as

$$\begin{aligned} G_{11} &\xrightarrow{D} \frac{1}{G_{11}}, \quad G_{12} \xrightarrow{D} \frac{B_{21}}{G_{11}}, \\ B_{21} &\xrightarrow{D} \frac{G_{12}}{G_{11}}, \quad G_{22} \xrightarrow{D} G_{22} - \frac{G_{12}^2 - B_{21}^2}{G_{11}}, \end{aligned} \quad (26)$$

which may be recognized as the factorized duality.

Other discrete symmetries are [4]

$$W: \quad (\tau, \rho) \leftrightarrow (\tau, -\bar{\rho}), \quad \tilde{\phi} \rightarrow \tilde{\phi}, \quad (27)$$

as well as

$$R: \quad (\tau, \rho) \leftrightarrow (-\bar{\tau}, \rho), \quad \tilde{\phi} \rightarrow \tilde{\phi}, \quad (28)$$

with $R = DWDW$. The W, R discrete symmetries leave invariant the fields G_{ij} , G_{11} , G_{22} and ϕ while

$$\begin{aligned} G_{12} &\xrightarrow{W} G_{12} \quad , \quad B_{21} \xrightarrow{W} -B_{21} \, , \\ G_{12} &\xrightarrow{R} -G_{12} \quad , \quad B_{21} \xrightarrow{R} -B_{21} \, . \end{aligned} \quad (29)$$

Let us note that there exists another Lagrangian which leads to the same equations as (23). It can be constructed by using the fact that in 3-dimensions, two-forms like $F_{\mu\nu}$ and $H_{\mu\nu}$ can be written as

$$\begin{aligned} F^{\mu\nu} &= \frac{1}{\sqrt{3}} \frac{\eta^{\mu\nu\rho}}{\sqrt{-\gamma}} F_\rho \, , \\ H^{\mu\nu} &= \frac{1}{\sqrt{3}} \frac{\eta^{\mu\nu\rho}}{\sqrt{-\gamma}} H_\rho \, . \end{aligned} \quad (30)$$

The Bianchi identities for $F_{\mu\nu}$, $H_{\mu\nu}$ then imply

$$\nabla_\mu F^\mu = 0 \quad , \quad \nabla_\mu H^\mu = 0. \quad (31)$$

Thus, we may express (13) as

$$\begin{aligned} \mathcal{L}^* &= \sqrt{-\gamma} G_{11}^{1/2} e^\phi \left(R + \frac{1}{2} G_{11} F_\mu F^\mu + G_{11}^{-1/2} \epsilon^{-\phi} \psi \nabla_\mu F^\mu \right. \\ &\quad \left. + \frac{1}{2} \frac{1}{G_{11}} H_\mu H^\mu + G_{11}^{-1/2} \epsilon^{-\phi} b \nabla_\mu H^\mu + \partial_\mu \phi \partial^\mu \phi \right) , \end{aligned} \quad (32)$$

where the constraints (31) have been taken into account by employing the auxiliary fields (b, ψ) . The equations of motions for the H_μ , F_μ give

$$\begin{aligned} F_\mu &= G_{11}^{-3/2} e^{-\phi} \partial_\mu \psi \, , \\ H_\mu &= G_{11}^{1/2} e^{-\phi} \partial_\mu b \, , \end{aligned} \quad (33)$$

so that \mathcal{L}^* is written as

$$\begin{aligned} \mathcal{L}^* &= \sqrt{-\gamma} G_{11}^{1/2} e^\phi \left(R(\gamma) - \frac{1}{2} \frac{1}{G_{11}^2} e^{-2\phi} \partial_\mu \psi \partial^\mu \psi \right. \\ &\quad \left. - \frac{1}{2} e^{-2\phi} \partial_\mu b \partial^\mu b + \partial_\mu \phi \partial^\mu \phi \right) . \end{aligned} \quad (34)$$

If we further reduce it with respect to ξ_2 , we get

$$\begin{aligned} \mathcal{L}^* &= \sqrt{-G^{(2)}} G_{11}^{1/2} V^{1/2} e^\phi \left(R(G^{(2)}) + \frac{1}{2} \frac{\partial V}{V} \frac{\partial G_{11}}{G_{11}} - \frac{1}{2} \frac{1}{G_{11}^2} e^{-2\phi} (\partial\psi)^2 \right. \\ &\quad \left. + \frac{1}{2} e^{-2\phi} (\partial b)^2 + (\partial\phi)^2 \right) . \end{aligned} \quad (35)$$

The two Lagrangians \mathcal{L} , \mathcal{L}^* given by (20) (or (23)) and (35), respectively lead to the same equations of motions. \mathcal{L} is invariant under $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ while the symmetries of \mathcal{L}^* are less obvious. In order the invariance properties of both \mathcal{L} , \mathcal{L}^* to become transparent, we adapt the following parametrization

$$G_{11} = e^{-\phi} \sigma \quad , \quad V = e^{-\phi} \frac{\mu^2}{\sigma} \quad (36)$$

$$G_{ij} = e^{-\phi} \frac{\lambda^2}{\sigma} \eta_{ij} \quad , \quad (37)$$

where $\eta_{ij} = (-1, 1)$. The metric (17) is then written as

$$ds^2 = e^{-\phi} \sigma (d\theta_1 + A d\theta_2)^2 + e^{-\phi} \frac{1}{\sigma} (\mu^2 d\theta_2^2 + \lambda^2 \eta_{ij} dx^i dx^j) . \quad (38)$$

As a result, \mathcal{L} , \mathcal{L}^* turn out to be

$$\begin{aligned} \mathcal{L} = & \mu \left(2\partial \ln \mu \partial \left(\frac{e^{-\phi/2} \lambda \mu^{1/2}}{\sigma^{1/2}} \right) - \frac{1}{2} \frac{\sigma^2}{\mu^2} (\partial A)^2 - \frac{1}{2} \left(\partial \ln \frac{\sigma}{\mu} \right)^2 \right. \\ & \left. - \frac{1}{2} \frac{e^{2\phi}}{\mu^2} (\partial B)^2 - \frac{1}{2} (\partial \ln e^{-\phi} \mu)^2 \right) , \end{aligned} \quad (39)$$

and

$$\mathcal{L}^* = \mu \left(2\partial \mu \partial \ln \lambda - \frac{1}{2} \frac{1}{\sigma^2} (\partial \sigma)^2 - \frac{1}{2} \frac{1}{\sigma^2} (\partial \psi)^2 - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} e^{-2\phi} (\partial b)^2 \right) . \quad (40)$$

Note that (A, ψ) and (B, b) are related through the relations

$$\partial_i A = -\frac{1}{\sqrt{3}} \varepsilon_{ij} \frac{\mu}{\sigma^2} \eta^{jk} \partial_k \psi , \quad (41)$$

$$\partial_i B = -\frac{1}{\sqrt{3}} \varepsilon_{ij} e^{-2\phi} \mu \eta^{jk} \partial_k b , \quad (42)$$

where $\varepsilon_{12} = 1$ is the antisymmetric symbol in two-dimensions.

Let us now define, in addition to the (τ, ρ) fields given in eqs. (21,22), the complex fields (S, Σ)

$$S = b + i e^\phi \quad , \quad \Sigma = \psi + i \sigma . \quad (43)$$

Then \mathcal{L} , \mathcal{L}^* may be expressed as

$$\mathcal{L} = \mu \left(2\partial \ln \mu \partial \left(\frac{e^{-\phi/2} \lambda \mu^{1/2}}{\sigma^{1/2}} \right) + 2 \frac{\partial \tau \partial \bar{\tau}}{(\tau - \bar{\tau})^2} + 2 \frac{\partial \rho \partial \bar{\rho}}{(\rho - \bar{\rho})^2} \right) \quad (44)$$

$$\mathcal{L}^* = \mu \left(2\partial \mu \partial \ln \lambda + 2 \frac{\partial S \partial \bar{S}}{(S - \bar{S})^2} + 2 \frac{\partial \Sigma \partial \bar{\Sigma}}{(\Sigma - \bar{\Sigma})^2} \right) . \quad (45)$$

Thus, there exist four $SL(2, \mathbb{R})/U(1)$ -sigma models, \mathcal{L} is invariant under the $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ transformations (24) and \mathcal{L}^* is invariant under

$$S \rightarrow \frac{kS + m}{nS + \ell} \quad , \quad \Sigma \rightarrow \frac{\kappa\Sigma + \eta}{\nu\Sigma + \theta} . \quad (46)$$

These transformation do not affect μ . There also exist discrete Z_2 transformations, besides those that have already been noticed in eqs. (25,27,28), namely

$$D' : (S, \Sigma) \leftrightarrow (\Sigma, S) \quad (47)$$

$$W' : (S, \Sigma) \leftrightarrow (S, -\bar{\Sigma}) \quad (48)$$

$$R' : (S, \Sigma) \leftrightarrow (-\bar{S}, \Sigma) . \quad (49)$$

Moreover, the transformations

$$N : (\tau, \rho) \leftrightarrow (S, \Sigma) \quad , \quad \lambda \leftrightarrow e^{-\phi/2} \frac{\mu^{1/2}}{\sigma^{1/2}} \lambda , \quad (50)$$

$$N' : (\tau, \rho) \leftrightarrow (\Sigma, S) \quad , \quad \lambda \leftrightarrow e^{-\phi/2} \frac{\mu^{1/2}}{\sigma^{1/2}} \lambda , \quad (51)$$

identify the two Lagrangians and thus, may be considered as the string counterpart of the Kramer-Neugebauer symmetry [19]. Note that $\mathcal{L}, \mathcal{L}^*$ may also be written as

$$\mathcal{L} = \mu \left(2\partial \ln \mu \partial \left(\frac{e^{-\phi/2} \lambda \mu^{1/2}}{\sigma^{1/2}} \right) - \frac{1}{4} Tr(h_1^{-1} \partial h_1)^2 - \frac{1}{4} Tr(h_2^{-1} \partial h_2)^2 \right) \quad (52)$$

$$\mathcal{L}^* = \mu \left(2\partial \mu \partial \lambda - \frac{1}{4} Tr(g_1^{-1} \partial g_1)^2 - \frac{1}{4} Tr(g_2^{-1} \partial g_2)^2 \right) . \quad (53)$$

where the 2×2 matrices h_1, h_2, g_1 and g_2 are

$$h_1 = \begin{pmatrix} \frac{\sigma}{\mu} & \frac{\sigma}{\mu} A \\ \frac{\sigma}{\mu} A & \frac{\sigma}{\mu} A^2 + \frac{\mu}{\sigma} \end{pmatrix} , \quad h_2 = \begin{pmatrix} \frac{e^\phi}{\mu} & \frac{e^\phi}{\mu} B \\ \frac{e^\phi}{\mu} B & \frac{e^\phi}{\mu} B^2 + \frac{\mu}{e^\phi} \end{pmatrix} , \quad (54)$$

$$g_1 = \begin{pmatrix} \frac{1}{\sigma} & \frac{1}{\sigma} \psi \\ \frac{1}{\sigma} \psi & \frac{1}{\sigma} \psi^2 + \sigma \end{pmatrix} , \quad g_2 = \begin{pmatrix} e^{-\phi} & e^{-\phi} b \\ e^{\phi} b & e^{\phi} b^2 + e^{-\phi} \end{pmatrix} . \quad (55)$$

The Lagrangian \mathcal{L} is invariant under the infinitesimal transformations

$$\begin{aligned} \delta\sigma &= \sqrt{2} \frac{1}{\sigma} A \epsilon_1^+ - 2\epsilon_1^0 \quad , \quad \delta A = -\frac{1}{\sqrt{2}} \left(\frac{\sigma^2}{\mu^2} - A^2 \right) \epsilon_1^+ - 2A \epsilon_1^0 + \sqrt{2} \epsilon_1^- , \\ \delta\phi &= -\sqrt{2} B \epsilon_2^+ + 2\epsilon_2^0 \quad , \quad \delta B = -\frac{1}{\sqrt{2}} \left(\frac{e^{2\phi}}{\mu^2} - B^2 \right) \epsilon_2^+ - 2B \epsilon_2^0 + \sqrt{2} \epsilon_2^- , \end{aligned} \quad (56)$$

while \mathcal{L}^* is invariant under

$$\begin{aligned} \delta\sigma &= -\sqrt{2} \psi \sigma \epsilon_3^+ + 2\sigma \epsilon_3^0 \quad , \quad \delta\psi = -\frac{1}{\sqrt{2}} \left(\frac{1}{\sigma^2} - \psi^2 \right) \epsilon_3^+ - 2\psi \epsilon_3^0 + \sqrt{2} \epsilon_3^- , \\ \delta\phi &= \sqrt{2} b \epsilon_4^+ - 2\epsilon_4^0 \quad , \quad \delta b = -\frac{1}{\sqrt{2}} \left(e^{2\phi} - b^2 \right) \epsilon_4^+ - 2b \epsilon_4^0 + \sqrt{2} \epsilon_4^- . \end{aligned} \quad (57)$$

The above infinitesimal transformations are generated by a set of four Killing vectors ($\mathbf{K}_a^{(i)}$, $a = 1, 2, 3$, $i = 1, 2, 3, 4$) which can easily be written down by recalling that the metric

$$ds^2 = dx^2 + e^{2x} dy^2 \quad (58)$$

has a three-parameter group of isometries generated by

$$\begin{aligned} K_+ &= -\sqrt{2}y\partial_x - \frac{1}{\sqrt{2}}(e^{-2x} - y^2)\partial_y, \\ K_0 &= 2(\partial_x - y\partial_y), \\ K_- &= \sqrt{2}\partial_y, \end{aligned} \quad (59)$$

which satisfy the $SL(2)$ commutation relations

$$[K_+, K_0] = 2K_+, [K_-, K_0] = -2K_-, [K_-, K_+] = -K_0. \quad (60)$$

Among these Killing vectors, let us consider $K_0^{(3)}$ which scales both ψ and σ as

$$K_0^{(3)} : (\psi, \sigma) \rightarrow (\alpha\psi, \alpha\sigma). \quad (61)$$

In view of eq. (41), A is also scaled as

$$A \rightarrow \frac{1}{\alpha}A, \quad (62)$$

so that (A, σ) is transformed into $(\frac{1}{\alpha}A, \alpha\sigma)$ which is generated by $-K_0^{(1)}$. However, \mathcal{L} is not invariant unless we also scale the conformal factor λ as $\sqrt{\alpha}\lambda$. Let us denote the generator of constant Weyl transformations by k . Then we have the relation

$$K_0^{(1)} + K_0^{(3)} = k. \quad (63)$$

In the same way, one may see that $K_0^{(2)}$, $K_0^{(4)}$ which transform (B, ϕ) and (b, ϕ) as $(e^{-\alpha}B, \phi + \alpha)$, $(e^\alpha, \phi + \alpha)$ respectively satisfy

$$K_0^{(2)} + K_0^{(4)} = k. \quad (64)$$

As a result, the algebra turns out to be

$$\begin{aligned} [K_+^{(1)}, K_0^{(1)}] &= 2K_+^{(1)}, & [K_-^{(1)}, K_0^{(1)}] &= -2K_-^{(1)}, & [K_-^{(1)}, K_+^{(1)}] &= K_0^{(1)}, \\ [K_+^{(2)}, K_0^{(2)}] &= 2K_+^{(2)}, & [K_-^{(2)}, K_0^{(2)}] &= -2K_-^{(2)}, & [K_-^{(2)}, K_+^{(2)}] &= K_0^{(2)}, \\ [K_+^{(3)}, k - K_0^{(1)}] &= 2K_+^{(3)}, & [K_-^{(3)}, k - K_0^{(1)}] &= -2K_-^{(3)}, & [K_-^{(3)}, K_+^{(3)}] &= k - K_0^{(1)}, \\ [K_+^{(4)}, k - K_0^{(2)}] &= 2K_+^{(4)}, & [K_-^{(4)}, k - K_0^{(2)}] &= -2K_-^{(4)}, & [K_-^{(4)}, K_+^{(4)}] &= k - K_0^{(2)} \end{aligned} \quad (65)$$

If we define the generators (h_i, k_i, f_i) by

$$h_i = K_0^{(i)}, \quad f_i = K_+^{(i)}, \quad e_i = K_-^{(i)}, \quad (66)$$

then the algebra (65) may be written as

$$\begin{aligned} [h_i, h_j] &= 0, \\ [h_i, e_j] &= A_{ij} e_j, \\ [h_i, f_j] &= -A_{ij}, \\ [e_i, f_j] &= \delta_{ij} h_j, \end{aligned} \quad (67)$$

where the Cartan matrix A_{ij} is

$$A_{ij} = \begin{pmatrix} a_{ij} & 0 \\ 0 & a_{ij} \end{pmatrix}, \quad a_{ij} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \quad (68)$$

In addition, one may verify the Serre relation

$$(ade_i)^{1-A_{ij}}(e_j) = 0, \quad (adf_i)^{1-A_{ij}}(f_j) = 0. \quad (69)$$

As a result, the algebra generated by the successive applications of the transformations (56,57) is the affine Kac-Moody algebra $\hat{o}(2, 2)$ with a central term corresponding to constant Weyl rescalings of the 2-dimensional background metric. The central term survives in higher dimensions as well, since its emergence is related to the existence of two alternative effective Lagrangians after reducing the 3-dimensional theory down to two dimensions over an abelian isometry. It is the interplay of the symmetries of these Lagrangians which produce the Kac-Moody algebra.

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